3 Majorana particles (10 points)

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The charge carriers in an electrical conductor can be negatively charged electrons or positively charged holes. An electron can annihilate a hole. We therefore call the hole the anti-particle of the electron. A Majorana particle is its own anti-particle, so two Majorana particles can only exist if they are are spacially separated. In experiments, two Majorana particles are contained at both ends of a thin, superconducting wire.

Question 1: Give an educated estimation of the charge of the Majorana particle, expressed in multiples of the electron charge $-e$.

The quantum mechanical wave functions of the electron $\psi_e(x)$ and of the hole $\psi_h(x)$ in the thin wire (along the x-axis) satisfy at a given energy E the following two coupled differential equations:

$$\begin{cases} -eV(x)\psi_e(x) - i\hbar v \frac{d}{dx}\psi_h(x) = E\psi_e(x) \\ +eV(x)\psi_h(x) - i\hbar v \frac{d}{dx}\psi_e(x) = E\psi_h(x) \end{cases} \tag{3.1}$$

The wave velocity is v, and $V(x)$ is the electric potential in the wire. A solution of equation 3.1 descripes a new type of particle, consisting of an electron and a hole.

Question 2: Prove the particle-anti-particle symmetry: if there is a solution of 3.1 with energy E, then there also is a solution with energy $-E$.

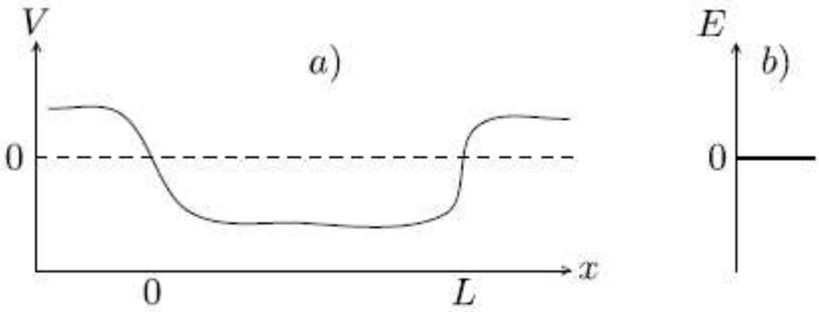

Figure 3.1: *a)* The electric potential $V(x)$ along the wire. *b)* The energy spectrum of the wire in the limit $L \to \infty$.

To find a Majorana particle, we look for a solution of 3.1 with energy $E = 0$. This composite particle is its own anti-particle. The potential $V(x)$ roughly looks like 3.1, but the exact shape is not important, as long as $V(x)$ is negative inside the wire, and positive outside of it. The left end of the wire is at $x = 0$, the right end at $x = L$. We will simplify things a little by considering the limit $L \to \infty$.

Question 3: Prove that the wave function of the Majorana particle is given by

$$\psi_e(x) = C \cdot \exp\left(\frac{se}{\hbar v}\int_0^x V(x')dx'\right), \quad \psi_h(x) = is\psi_e(x),$$

where C is some constant, and s equals $+1$ or -1. Which value of s corresponds to the Majorana particle at the left end of the wire?

Question 4: We have now found a pair of Majorana particles for $L \to \infty$ at $E = 0$. The energy spectrum of the wire therefore looks like figure 3.1b. Sketch the energy spectrum in the case where L is large, but not infinitely large.